# Linear angle preservers of Hilbert bundles

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Let $T : H \to K$ be a linear map. Utilizing the polar identities,

$$4\langle Tx, Ty\rangle = \sum_{k=0}^{3} i^k \|Tx + i^k Ty\|^2 \quad \text{for the complex case, or}$$

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we see that $T$ is an isometry, i.e., $\|Tx\| = \|x\|$, $\forall x \in H$, exactly when $T$ preserves inner products, i.e., $\langle Tx, Ty\rangle = \langle x, y\rangle$, $\forall x, y \in H$.

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In this case, we must have $\theta = \phi$.

**Proof for $\theta = \phi = \pi/2$.** Let $x, y \in H$ with $\|x\| = \|y\| = 1$. Let $\lambda$ be the unimodular scalar such that $\lambda\langle x, y\rangle = |\langle x, y\rangle|$.
Then $\langle \lambda x + y, \lambda x - y\rangle = 0$. It then follows $\langle \lambda Tx + Ty, \lambda Tx - Ty\rangle = 0$.
Consequently, $\|Tx\|^2 - \|Ty\|^2 - \lambda\langle Tx, Ty\rangle + \overline{\lambda\langle Tx, Ty\rangle} = 0$.
Equating the real parts, we see that $\|Tx\| = \|Ty\| = \alpha > 0$.
Consequently, $\frac{1}{\alpha}T$ is an isometry.
The real case is even easier. □

A (real or complex) Banach bundle over a locally compact Hausdorff space $X$ is a pair $(B_X, \pi_X)$, in which $B_X$ is a topological space and $\pi_X$ is a continuous open surjective map from $B_X$ onto $X$ satisfying the following conditions.

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(3) If $x \in X$ and $\{b_i\}$ is any net in $B_X$ such that $\|b_i\| \to 0$ and $\pi(b_i) \to x$ in $X$, then $b_i \to 0_x$ (the zero element of $B_x$) in $B_X$.

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**Consider the Banach space of continuous vector sections:**

$$C_0(X; B_X) = \{f : X \to B_X \mid f \text{ is continuous such that } f(x) \in B_x, \forall x \in X \text{ and } \lim_{x\to\infty} \|f(x)\| = 0\}.$$

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For a Hilbert bundle $(H_X, \pi_X)$ over a locally compact space $X$, the Banach space $C_0(X; H_X)$ of continuous vector sections forms a Hilbert $C_0(X)$-module with module multiplication

$$(\alpha f)(x) = \alpha(x) f(x), \quad \forall \alpha \in C_0(X), \forall f \in C_0(X; H_X), \forall x \in X,$$

and the $C_0(X)$-inner product

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A linear map $T : \mathcal{H} \to \mathcal{K}$ between Hilbert $\mathcal{A}$-modules is a module map if

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Suppose that $u, v \in C(X)$ with $0 \leq u, v < 1$ and

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Let $T : \mathcal{H} \to \mathcal{K}$ be a bijective complex linear map from a complex Hilbert $C_0(X)$-module $\mathcal{H}$ into a complex Hilbert $C_0(Y)$-module $\mathcal{K}$. We say that $T$ is quasi-local if

$$\operatorname{supp} T^{-1}(T(f)\beta') \subseteq \operatorname{supp} f, \quad \forall f \in \mathcal{H}, \forall \beta' \in C_0(Y).$$

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**Theorem 5.** Let $T : \mathcal{H} \to \mathcal{K}$ be a bijective complex linear map from a full complex Hilbert $C_0(X)$-module $\mathcal{H}$ onto a full complex Hilbert $C_0(Y)$-module $\mathcal{K}$, such that both $T$ and $T^{-1}$ are quasi-local.

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